

DOMINATION PROBLEM FOR NARROW ORTHOGONALLY ADDITIVE OPERATORS

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ABSTRACT. The “Up-and-down” theorem which describes the structure of the Boolean algebra of fragments of a linear positive operator is the well known result of the operator theory. We prove an analog of this theorem for a positive abstract Uryson operator defined on a vector lattice and taking values in a Dedekind complete vector lattice. This result we apply to prove that for an order narrow positive abstract Uryson operator T from a vector lattice E to a Dedekind complete vector lattice F , every abstract Uryson operator $S : E \rightarrow F$, such that $0 \leq S \leq T$ is also order narrow.

1. INTRODUCTION

Today the theory of narrow operators is an active area of Functional Analysis (see the recent monograph [23]). Lately the concept of the narrowness was generalized to the setting of orthogonally additive operators in vector lattices [21]. The aim of this article is to continue the investigation of order narrow orthogonally additive operators and to consider the domination problem for this class of operators.¹

2. PRELIMINARIES

The goal of this section is to introduce some basic definitions and facts. General information on vector lattices and Boolean algebras the reader can find in the books [2, 8, 9, 14].

Let E be a vector lattice. A net $(x_\alpha)_{\alpha \in \Lambda}$ in E *order converges* to an element $x \in E$ (notation $x_\alpha \xrightarrow{(o)} x$) if there exists a net $(u_\alpha)_{\alpha \in \Lambda}$ in E_+ such that $u_\alpha \downarrow 0$ and $|x_\beta - x| \leq u_\beta$ for all $\beta \in \Lambda$. The equality $x = \bigsqcup_{i=1}^n x_i$ means that $x = \sum_{i=1}^n x_i$ and $x_i \perp x_j$ if $i \neq j$. An element y of E is called a *fragment* (in another terminology, a *component*) of an element $x \in E$, provided $y \perp (x - y)$. The notation $y \sqsubseteq x$ means that y is a fragment of x . Two fragments x_1, x_2

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of x are called *mutually complemented* or *MC*, in short, if $x = x_1 \sqcup x_2$. If E is a vector lattice and $e \in E$ then by \mathcal{F}_e we denote the set of all fragments of e .

An element e of a vector lattice E is called a *projection element* if the band generated by e is a projection band. A vector lattice E is said to have the *principal projection property* if every element of E is a projection element. For instance, every Dedekind σ -complete vector lattice has the principal projection property.

Definition 2.1. Let E be a vector lattice, and let F be a real linear space. An operator $T : E \rightarrow F$ is called *orthogonally additive* if $T(x + y) = T(x) + T(y)$ whenever $x, y \in E$ are disjoint.

It follows from the definition that $T(0) = 0$. It is immediate that the set of all orthogonally additive operators is a real vector space with respect to the natural linear operations.

Definition 2.2. Let E and F be vector lattices. An orthogonally additive operator $T : E \rightarrow F$ is called:

- *positive* if $Tx \geq 0$ holds in F for all $x \in E$;
- *order bounded* if T maps order bounded sets in E to order bounded sets in F .

An orthogonally additive, order bounded operator $T : E \rightarrow F$ is called an *abstract Uryson operator*. This class of operators was introduced and studied in 1990 by Mazón and Segura de León [16, 17], and then extended to lattice-normed spaces by Kusraev and the second named author [11, 12, 20]. Currently orthogonally additive operators are an active area of investigations [4, 5, 6, 21, 22].

For example, any linear operator $T \in L_+(E, F)$ defines a positive abstract Uryson operator by $G(f) = T|f|$ for each $f \in E$. Observe that if $T : E \rightarrow F$ is a positive orthogonally additive operator and $x \in E$ is such that $T(x) \neq 0$ then $T(-x) \neq -T(x)$, because otherwise both $T(x) \geq 0$ and $T(-x) \geq 0$ imply $T(x) = 0$. So, the above notion of positivity is far from the usual positivity of a linear operator: the only linear operator which is positive in the above sense is zero. A positive orthogonally additive operator need not be order bounded. Consider, for example, the real function $T : \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$T(x) = \begin{cases} \frac{1}{x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

The set of all abstract Uryson operators from E to F we denote by $\mathcal{U}(E, F)$. Consider some examples. The most famous one is the nonlinear integral Uryson operator.

Example 2.3. Let (A, Σ, μ) and (B, Ξ, ν) be σ -finite complete measure spaces, and let $(A \times B, \mu \times \nu)$ denote the completion of their product measure space. Let $K : A \times B \times \mathbb{R} \rightarrow \mathbb{R}$ be a function satisfying the following conditions²:

- (C₀) $K(s, t, 0) = 0$ for $\mu \times \nu$ -almost all $(s, t) \in A \times B$;
- (C₁) $K(\cdot, \cdot, r)$ is $\mu \times \nu$ -measurable for all $r \in \mathbb{R}$;
- (C₂) $K(s, t, \cdot)$ is continuous on \mathbb{R} for $\mu \times \nu$ -almost all $(s, t) \in A \times B$.

Given $f \in L_0(B, \Xi, \nu)$, the function $|K(s, \cdot, f(\cdot))|$ is ν -measurable for μ -almost all $s \in A$ and $h_f(s) := \int_B |K(s, t, f(t))| d\nu(t)$ is a well defined and μ -measurable function. Since the function h_f can be infinite on a set of positive measure, we define

$$\text{Dom}_B(K) := \{f \in L_0(\nu) : h_f \in L_0(\mu)\}.$$

Then we define an operator $T : \text{Dom}_B(K) \rightarrow L_0(\mu)$ by setting

$$(Tf)(s) := \int_B K(s, t, f(t)) d\nu(t) \quad \mu - \text{a.e.} \quad (\star)$$

Let E and F be order ideals in $L_0(\nu)$ and $L_0(\mu)$ respectively, K a function satisfying (C₀)-(C₂). Then (\star) defines an orthogonally additive order bounded integral operator acting from E to F if $E \subseteq \text{Dom}_B(K)$ and $T(E) \subseteq F$.

Example 2.4. We consider the vector space \mathbb{R}^m , $m \in \mathbb{N}$ as a vector lattice with the coordinate-wise order: for any $x, y \in \mathbb{R}^m$ we set $x \leq y$ provided $e_i^*(x) \leq e_i^*(y)$ for all $i = 1, \dots, m$, where $(e_i^*)_{i=1}^m$ are the coordinate functionals on \mathbb{R}^m . Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$. Then $T \in \mathcal{U}(\mathbb{R}^n, \mathbb{R}^m)$ if and only if there are real functions $T_{i,j} : \mathbb{R} \rightarrow \mathbb{R}$, $1 \leq i \leq m$, $1 \leq j \leq n$ satisfying $T_{i,j}(0) = 0$ such that

$$e_i^*(T(x_1, \dots, x_n)) = \sum_{j=1}^n T_{i,j}(x_j),$$

In this case we write $T = (T_{i,j})$.

Example 2.5. Let $T : l^2 \rightarrow \mathbb{R}$ be the operator defined by

$$T(x_1, \dots, x_n, \dots) = \sum_{n \in I_x} n(|x_n| - 1)$$

where $I_x := \{n \in \mathbb{N} : |x_n| \geq 1\}$. It is not difficult to check that T is a positive abstract Uryson operator.

Example 2.6. Let (Ω, Σ, μ) be a measure space, E a sublattice of the vector lattice $L_0(\mu)$ of all equivalence classes of Σ -measurable functions $x : \Omega \rightarrow \mathbb{R}$, F a vector lattice and $\nu : \Sigma \rightarrow F$ a finitely additive measure. Then the map $T : E \rightarrow F$ given by $T(x) = \nu(\text{supp } x)$ for any $x \in E$, is an abstract Uryson operator which is positive if and only if ν is positive.

Consider the following order in $\mathcal{U}(E, F)$: $S \leq T$ whenever $T - S$ is a positive operator. Then $\mathcal{U}(E, F)$ becomes an ordered vector space. If a vector lattice F is Dedekind complete we have the following theorem.

²(C₁) and (C₂) are called the Carathéodory conditions

Theorem 2.7. ([16], Theorem 3.2). *Let E and F be a vector lattices, F Dedekind complete. Then $\mathcal{U}(E, F)$ is a Dedekind complete vector lattice. Moreover for $S, T \in \mathcal{U}(E, F)$ and for $f \in E$ following hold*

- (1) $(T \vee S)(f) := \sup\{Tg_1 + Sg_2 : f = g_1 \sqcup g_2\}.$
- (2) $(T \wedge S)(f) := \inf\{Tg_1 + Sg_2 : f = g_1 \sqcup g_2\}.$
- (3) $(T)^+(f) := \sup\{Tg : g \sqsubseteq f\}.$
- (4) $(T)^-(f) := -\inf\{Tg : g; g \sqsubseteq f\}.$
- (5) $|Tf| \leq |T|(f).$

We follow [21] in the next definition.

Definition 2.8. Let E, F be vector lattices with E an atomless. An abstract Uryson operator $T : E \rightarrow F$ is called *order narrow* if for every $e \in E$ there exists a net of decompositions $e = f_\alpha \sqcup g_\alpha$ such that $(T(f_\alpha) - T(g_\alpha)) \xrightarrow{(o)} 0$.

It is a worth noting that linear order narrow operators were firstly introduced by Maslyuchenko, Mykhaylyuk and Popov in [15]. Lately, in setting of lattice-normed spaces linear order narrow operators were investigated by the author in [19].

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3. THE BOOLEAN ALGEBRA OF FRAGMENTS OF A POSITIVE URYSON OPERATOR

Let E, F be vector lattices with F Dedekind complete and $T \in \mathcal{U}_+(E, F)$. The purpose of this section is to describe the fragments of T . That is

$$\mathcal{F}_T = \{S \in \mathcal{U}_+(E, F) : S \wedge (T - S) = 0\}.$$

Like in the linear case we consider elementary fragments. For a subset \mathcal{A} of a vector lattice W we employ the following notation:

$$\mathcal{A}^\uparrow = \{x \in W : \exists \text{ a sequence } (x_n) \subset \mathcal{A} \text{ with } x_n \uparrow x\};$$

$$\mathcal{A}^\downarrow = \{x \in W : \exists \text{ a net } (x_\alpha) \subset \mathcal{A} \text{ with } x_\alpha \downarrow x\}.$$

The meanings of \mathcal{A}^\uparrow and \mathcal{A}^\downarrow are analogous. As usual, we also write

$$\mathcal{A}^{\uparrow\downarrow} = (\mathcal{A}^\downarrow)^\uparrow; \mathcal{A}^{\downarrow\uparrow} = ((\mathcal{A}^\uparrow)^\downarrow)^\uparrow.$$

It is clear that $\mathcal{A}^{\uparrow\downarrow} = \mathcal{A}^\downarrow$, $\mathcal{A}^{\downarrow\uparrow} = \mathcal{A}^\uparrow$. Consider a positive abstract Uryson operator $T : E \rightarrow F$, where F is Dedekind complete. Since \mathcal{F}_T is a Boolean algebra, it is closed under finite suprema and infima. In particular, all “ups and downs” of \mathcal{F}_T are likewise closed under finite suprema and infima, and therefore they are also directed upward and, respectively, downward.

Definition 3.1. A subset D of a vector lattice E is called a *lateral ideal* if the following conditions hold

- (1) if $x \in D$ then $y \in D$ for every $y \in \mathcal{F}_x$;
- (2) if $x, y \in D$, $x \perp y$ then $x + y \in D$.

Consider some examples.

Example 3.2. *Let E be a vector lattice. Every order ideal in E is a lateral ideal.*

Example 3.3. *Let E, F be vector lattices and $T \in \mathcal{U}_+(E, F)$. Then $\mathcal{N}_T := \{e \in E : T(e) = 0\}$ is a lateral ideal.*

The following example is important for further considerations.

Lemma 3.4. ([4], Lemma 3.5). *Let E be a vector lattice and $x \in E$. Then \mathcal{F}_x is a lateral ideal.*

Let $T \in \mathcal{U}_+(E, F)$ and $D \subset E$ be a lateral ideal. Then for every $x \in E$, we define a map $\pi^D T : E \rightarrow F_+$ by the following formula

$$(3.1) \quad \pi^D T(x) = \sup\{Ty : y \in \mathcal{F}_x \cap D\}.$$

Lemma 3.5. ([4], Lemma 3.6). *Let E, F be vector lattices with F Dedekind complete, $\rho \in \mathfrak{B}(F)$, $T \in \mathcal{U}_+(E, F)$ and D be a lateral ideal. Then $\pi^D T$ is a positive abstract Uryson operator and $\rho\pi^D T \in \mathcal{F}_T$.*

If $D = \mathcal{F}_x$ then the operator $\pi^D T$ is denoted by $\pi^x T$. Let F be a vector lattice. Recall that a family of mutually disjoint order projections $(\rho_\xi)_{\xi \in \Xi}$ on F is said to be *partition of unity* if $\bigvee_{\xi \in \Xi} (\rho_\xi)_{\xi \in \Xi} = Id_F$. Any fragment of

the form $\sum_{i=1}^n \rho_i \pi^{x_i} T$, $n \in \mathbb{N}$, where ρ_1, \dots, ρ_n is a finite family of mutually disjoint order projections in F , like in the linear case is called an *elementary fragment* T . The set of all elementary fragments of T we denote by \mathcal{A}_T .

For further considerations we need the following auxiliary proposition, which was proven by nonstandard methods.

Lemma 3.6 ([10], Proposition 5.2.7.2). *Let F be a Dedekind complete vector lattice with a weak order unit³ u and $(x_\lambda)_{\lambda \in \Lambda}$ be an order bounded net in F . Then the net $(x_\lambda)_{\lambda \in \Lambda}$ order converges to an element $x \in F$ if and only if for every $\varepsilon > 0$ there exists a partition of unity $(\rho_\lambda)_{\lambda \in \Lambda}$ such that*

$$\rho_\lambda |x_\beta - x| \leq \varepsilon u, \quad \beta \geq \lambda.$$

Remark 3.7. *Observe that every Dedekind complete vector lattice is an order dense ideal in some Dedekind complete vector lattice with a weak order unit ([24], Theorem 4.7.2).*

Lemma 3.8. *Let E, F be vector lattices, F be Dedekind complete and let \mathfrak{A} be the set of all weak order units in F . If operators $T, S \in \mathcal{U}_+(E, F)$ are disjoint, then for every $x \in E$, $u \in \mathfrak{A}$ and $\varepsilon > 0$ there exists a partition of unity $(\pi_\xi)_{\xi \in \Xi}$ in $\mathfrak{B}(F)$ and a family $(x_\xi)_{\xi \in \Xi}$ of fragments of x , such that*

$$\pi_\xi(Tx_\xi + S(x - x_\xi)) \leq \varepsilon u \quad \text{for all } \xi \in \Xi.$$

³An element $u \in F_+$ is a *weak order unit* if $\{u\}^{\perp\perp} = F$, i.e. except 0 there are no elements in F which are disjoint to u .

Proof. Take any $x \in E$. Denote by Ξ , the set of all pairs $\xi = (y, z) \in \mathcal{F}_x \times \mathcal{F}_x$ of mutually disjoint fragments of x , such that $y + z = x$. For any $\xi = (y, x - y) \in \Xi$ put $f_\xi = Ty + S(x - y)$. Due to formula (2) of Theorem 2.7 the disjointness of the operators S and T implies $\inf_{\xi \in \Xi} \{f_\xi\} = 0$. Denote

by Δ the collection of all finite subsets of Ξ ordered as usual by inclusion, i.e. $\alpha \leq \alpha'$ iff $\alpha \subset \alpha'$. Introduce a set $(y_\alpha)_{\alpha \in \Delta}$ of all infima of finitely many elements of the set $\{f_\xi : \xi \in \Xi\}$, i.e. if $\alpha \in \Delta$ is a finite set $\alpha = \{\xi_{\alpha_1}, \dots, \xi_{\alpha_n}\}$, where $\xi_{\alpha_k} \in \Xi$ for $k = 1, \dots, n$, then

$$y_\alpha = \bigwedge_{i=1}^n f_{\xi_{\alpha_i}}$$

The set $(y_\alpha)_{\alpha \in \Delta}$ is downwards directed and $\sigma\text{-}\lim_{\alpha \in \Delta} y_\alpha = 0$. By Proposition 3.6, for every $\varepsilon > 0$ and $u \in \mathfrak{A}$ there exists a partition of unity $(\rho_\alpha)_{\alpha \in \Delta}$ in $\mathfrak{B}(F)$ such that

$$\rho_\alpha(y_\alpha) \leq \varepsilon u \text{ for all } \alpha \in \Delta.$$

In particular, $\rho_\alpha(f_\xi) < \varepsilon u$ if $\alpha = \xi$.

Identify now F with a vector sublattice of the Dedekind complete vector lattice $C_\infty(Q)$ of all extended real valued continuous functions on some extremally disconnected compact space Q (more exactly with its image under some vector lattice isomorphism), where the choosen weak order unit u is mapped onto the constant function $\mathbf{1}$ on Q (see [1], Theorem 3.35). Then the order projections $(\rho_\alpha)_{\alpha \in \Delta}$ (of the above partition of unity) are the multiplication operators in the space $C_\infty(Q)$ generated by the characteristic functions $\mathbf{1}_{Q_\alpha}$, respectively, where Q_α for all $\alpha \in \Delta$ are closed-open subsets of Q such that $Q = \bigcup_{\alpha \in \Delta} Q_\alpha$ and $Q_\alpha \cap Q_{\alpha'} = \emptyset$ for every $\alpha, \alpha' \in \Delta$, $\alpha \neq \alpha'$.

The supremum $\sup_{\alpha \in \Delta} \rho_\alpha$ is the identity operator I_F .

For $\alpha \in \Delta$ and $\xi \in \Xi$ define the set

$$A_\xi^\alpha = \{t \in Q_\alpha : f_\xi(t) < f_\beta(t), \beta \in \alpha, \beta \neq \xi\}$$

and denote by $\overline{A_\xi^\alpha}$ its closure in Q_α and, consequently in Q . So $\overline{A_\xi^\alpha}$ are closed-open subsets of Q for every $\alpha \in \Delta$, $\xi \in \Xi$ and, mutually disjoint if at least one index is different $\xi \neq \xi'$ or $\alpha \neq \alpha'$. Denote by ρ_ξ^α the multiplication operator generated by the characteristic function $\mathbf{1}_{\overline{A_\xi^\alpha}}$, i.e. $\rho_\xi^\alpha(f) = f \cdot \mathbf{1}_{\overline{A_\xi^\alpha}}$ for any function $f \in C_\infty(Q)$. It is clear that ρ_ξ^α is an order projection in $C_\infty(Q)$ and $\overline{A_\xi^\alpha} \subset Q_\alpha$ implies $\rho_\xi^\alpha \leq \rho_\alpha$. Hence $\rho_\xi^\alpha(f_\xi) \leq \varepsilon u$ for every $\xi \in \Xi$ and every $\alpha \in \Delta$. By what has been mentioned above the order projections ρ_ξ^α are mutually disjoint, whenever $\xi \neq \xi'$ or $\alpha \neq \alpha'$. Therefore, the order projections $\pi_\xi = \sup_{\alpha \in \Delta} \rho_\xi^\alpha$ and $\pi_{\xi'} = \sup_{\alpha \in \Delta} \rho_{\xi'}^\alpha$ are mutually disjoint as well. We show that the supremum of all π_ξ is the identity operator. By assuming the contrary there is a nonzero order projection γ which is disjoint to each projection π_ξ what causes its disjointness to each ρ_ξ^α and finally, γ is disjoint

to each ρ_α . This contradicts the fact that $(\rho_\alpha)_{\alpha \in \Delta}$ is a partition of unity. Thus $(\pi_\xi)_{\xi \in \Xi}$ is a partition of unity and

$$\pi_\xi(Tx_\xi + S(x - x_\xi)) \leq \varepsilon u \text{ for every } \xi \in \Xi.$$

■

Lemma 3.9. *Let E, F, \mathfrak{A}_F be the same as in the Lemma 3.8, $S, T \in \mathcal{U}_+(E, F)$. If $S \perp T$, then for every $x \in E$, $\varepsilon > 0$, $\mathbf{1} \in \mathfrak{A}_F$ there exists a partition of unity $(\rho_\xi)_{\xi \in \Xi}$ in $\mathfrak{B}(F)$, and a family $(x_\xi)_{\xi \in \Xi}$ of fragments of x such that $\rho_\xi \pi^{x_\xi} T(x) \leq \varepsilon \mathbf{1}$ and $\rho_\xi(S - \rho_\xi \pi^{x_\xi} S)x \leq \varepsilon \mathbf{1}$ for every $\xi \in \Xi$.*

Proof. Observe that for every $y \in \mathcal{F}_x$, $x \in E$ we have $\pi^y Tx = Ty$. Fix a weak order unit $\mathbf{1}$ and $\varepsilon > 0$. By Lemma 3.8 there exist a partition of unity $(\rho_\xi)_{\xi \in \Xi}$ in F , and a family $(x_\xi)_{\xi \in \Xi}$ of fragments of x such that

$$\rho_\xi(Tx_\xi + S(x - x_\xi)) \leq \varepsilon u \text{ for all } \xi \in \Xi.$$

Consequently, $\rho_\xi Tx_\xi = \rho_\xi \pi^{x_\xi} Tx \leq \varepsilon \mathbf{1}$ and

$$\rho_\xi S(x - x_\xi) = \rho_\xi Sx - \rho_\xi Sx_\xi = \rho_\xi(S - \rho_\xi \pi^{x_\xi} S)x \leq \varepsilon \mathbf{1}.$$

■

Lemma 3.10. *Let E, F, \mathfrak{A}_F be the same as in the Lemma 3.8, $T \in \mathcal{U}_+(E, F)$. If $S \in \mathcal{F}_T$ then for every $x \in E$, $\varepsilon > 0$, $\mathbf{1} \in \mathfrak{A}_F$ there exists a partition of unity $(\rho_\xi)_{\xi \in \Xi}$ in $\mathfrak{B}(F)$, and a family $(x_\xi)_{\xi \in \Xi}$ of fragments of x , such that $\rho_\xi |S - \rho_\xi \pi^{x_\xi} T|x \leq \varepsilon \mathbf{1}$ for every $\xi \in \Xi$.*

Proof. Using Lemma 3.9 we have

$$\begin{aligned} \rho_\xi |S - \rho_\xi \pi^{x_\xi} T|x &\leq \rho_\xi |S - \rho_\xi \pi^{x_\xi} S|x + \rho_\xi |\rho_\xi \pi^{x_\xi} S - \rho_\xi \pi^{x_\xi} T|x = \\ &= \rho_\xi |S - \rho_\xi \pi^{x_\xi} S|x + \rho_\xi |\rho_\xi \pi^{x_\xi} (T - S)|x \leq \varepsilon \mathbf{1}. \end{aligned}$$

■

Lemma 3.11. *Let E, F be the same as in Lemma 3.9, $T \in \mathcal{U}_+(E, F)$ and $S \in \mathcal{F}_T$. Then*

- (1) *for every $x \in E$, $\varepsilon > 0$, $\mathbf{1} \in \mathfrak{A}_F$ there exists $G_x \in \mathcal{A}_T^\uparrow$, so that $|S - G_x|x \leq \varepsilon \mathbf{1}$;*
- (2) *for every $x \in E$ there exists $R_x \in \mathcal{A}_T^{\uparrow \downarrow}$, so that $|S - R_x|x = 0$.*

Proof. Let us to prove (1). By Lemma 3.10 there exists a partition of unity $(\rho_\xi)_{\xi \in \Xi}$ in $\mathfrak{B}(F)$, and a family $(x_\xi)_{\xi \in \Xi}$ of fragments of x such that $\rho_\xi |S - \rho_\xi \pi^{x_\xi} T|x \leq \varepsilon \mathbf{1}$ for $\xi \in \Xi$. By Δ we denote the system of all finite subsets of Ξ . It is ordered by inclusion. Surely, Δ is a directed set. For every $\theta \in \Delta$ set $G_\theta = \sum_{\xi \in \theta} \rho_\xi \pi^{x_\xi} T$. The net $(G_\theta)_{\theta \in \Delta}$ is increasing. Let $G_x = \sup(G_\theta)_{\theta \in \Delta}$.

Then $G_x \in \mathcal{A}_T^\uparrow$ and we may write

$$\rho_\xi |S - G_\theta|x = \rho_\xi |S - \sum_{\xi \in \theta} \rho_\xi \pi^{x_\xi} T|x \leq \varepsilon \mathbf{1}$$

for every $\xi \in \Xi$ and every $\theta \geq \{\xi\}$. Therefore $\rho_\xi |S - G_x| x \leq \varepsilon \mathbf{1}$ for every $\xi \in \Xi$ and $|S - G_x| x \leq \varepsilon \mathbf{1}$.

Now we prove (2). Fix any $\mathbf{1} \in \mathfrak{A}_F$. For $\varepsilon_n = \frac{1}{2^n}$ there exists $G_x^n \in \mathcal{A}_T$ such that $|S - G_x^n| x \leq \frac{1}{2^n} \mathbf{1}$. Let $C_x^k = \bigvee_{n=k}^{\infty} G_x^n$ and $C_x^{k,i} = \bigvee_{n=k}^{n=k+i} G_x^n$. Since \mathcal{A}_T is a subalgebra of \mathcal{F}_T , one has $C_x^{k,i} \in \mathcal{A}_T^\uparrow$ and $C_x^{k,i} \uparrow C_x^k \in \mathcal{A}_T^{\uparrow\downarrow} = \mathcal{A}_T^\uparrow$. Then we have

$$\begin{aligned} |S - C_x^{k,i}| x &= \left| S - \bigvee_{n=k}^{n=k+i} G_x^n \right| x = \left| \bigwedge_{n=k}^{n=k+i} (S - G_x^n) \right| x \leq \\ &\leq \sum_{n=k}^{n=k+i} |S - G_x^n| x \leq \sum_{n=k}^{\infty} \frac{1}{2^n} \mathbf{1} \leq \frac{1}{2^{k-1}} \mathbf{1}. \end{aligned}$$

So we may write $|S - C_x^k| \leq \frac{1}{2^{k-1}} \mathbf{1}$. The sequence (C_x^k) is decreasing. Let $R_x = \inf C_x^k$. Then $R_x \in \mathcal{A}_T^{\uparrow\downarrow}$ and $|S - R_x| x = 0$. \blacksquare

Remark 3.12. Observe that $R_x y = 0$ for every y such that $\mathcal{F}_x \cap \mathcal{F}_y = 0$. Moreover, if $y \in \mathcal{F}_x$ and $|S - R_x| x = 0$ we can write $0 \leq |S - R_x| y \leq |S - R_x| x = 0$, and therefore $|S - R_x| y = 0$ for every $y \in \mathcal{F}_x$.

Lemma 3.13. Let E, F be the same as in Lemma 3.11, $T \in \mathcal{U}_+(E, F)$, $x \in E$ and $S \in \mathcal{F}_T$. Then there exists a $G \in \mathcal{A}_T^{\uparrow\downarrow}$ such that:

$$0 \leq G \leq S \text{ and } Gx = Sx.$$

Proof. Fix $x \in E$ and let

$$W := \{R \in \mathcal{A}_T^{\uparrow\downarrow} : |S - R| x = 0\}.$$

By Lemma 3.11 the set W is nonempty, and an easy argument shows that W is directed downward. Let $G = \inf\{W\}$. Clearly, $G \in \mathcal{A}_T^{\uparrow\downarrow} = \mathcal{A}_T^{\uparrow\downarrow}$, and hence $|S - G| x = 0$. We claim that $0 \leq G \leq S$. By Remark 3.12 $Gz = 0$ for every $z \in E$, such that $\mathcal{F}_z \cap \mathcal{F}_x = 0$ and we must prove $(G - S)^+ y = 0$ for every $y \in \mathcal{F}_x$. Now we may write

$$(G - S)^+ y \leq |R_x - S| y = |S - R_x| y = 0,$$

where $y \in \mathcal{F}_x$ and R_x is a some element of W . \blacksquare

The following theorem is the first main result of the article.

Theorem 3.14. Let E, F be vector lattices, F Dedekind complete, $T \in \mathcal{U}_+(E, F)$ and $S \in \mathcal{F}_T$. Then $S \in \mathcal{A}_T^{\uparrow\downarrow}$.

Proof. Let $S \in \mathcal{F}_T$ be fixed, and let

$$W = \{R \in \mathcal{A}_T^{\uparrow\downarrow} : 0 \leq R \leq S\}$$

Clearly, W is a directed set, and by Lemma 3.13 we know that $W \neq \emptyset$. Let $G = \sup\{W\}$, and remark that $0 \leq G \leq S$. On the other hand, if $x \in E$ is an arbitrary element of E , by Lemma 3.13 there exists some $R \in W$, such

that $0 \leq R \leq G \leq S$ and $Rx = Sx$. Thus $G = S$, $S \in \mathcal{A}_T^{\uparrow\downarrow\uparrow}$ and $\mathcal{F}_T = \mathcal{A}_T^{\uparrow\downarrow\uparrow}$.
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Remark that for linear positive operators the same theorem and its modifications were proved by de Pagter, Aliprantis and Burkinshaw, Kusraev and Strizhevski in [3, 13, 18].

4. DOMINATION PROBLEM FOR ABSTRACT URYSON NARROW OPERATORS

In this section we consider a domination problem for narrow abstract Uryson operators. In the classical sense, the domination problem can be stated as follows. Let E, F be vector lattices, $S, T : E \rightarrow F$ linear operators with $0 \leq S \leq T$. Let \mathcal{P} be some property of linear operators $R : E \rightarrow F$, so that $\mathcal{P}(R)$ means that R possesses \mathcal{P} . Does $\mathcal{P}(T)$ imply $\mathcal{P}(S)$?

Let E be a vector lattice and $x \in E_+$. The order ideal generated by x we denote by E_x . The following theorem is an important tool for further considerations.

Theorem 4.1. (*Freudenthal Spectral Theorem*) ([2], Theorem 2.8). *Let E be a vector lattice with the principal projection property and let $x \in E_+$. Then for every $y \in E_x$ there exists a sequence (u_n) of x -step functions satisfying $0 \leq y - u_n \leq \frac{1}{n}x$ for each n and $u_n \uparrow y$.*

The next theorem is the second main result of the article.

Theorem 4.2. *Let E, F be vector lattices, E atomless and with the principal projection property, F be Dedekind complete, and $T \in \mathcal{U}_+(E, F)$ be an order narrow operator. Then every operator $S \in \mathcal{U}_+(E, F)$, such that $0 \leq S \leq T$ is order narrow as well.*

For the proof we need an some auxiliary result. Let E, F be vector lattices, a family of operators $\{T_1, \dots, T_n\} \subset \mathcal{U}(E, F)$ is said to have *pairwise disjoint supports* if there exists a family of pairwise disjoint bands $E_1, \dots, E_n \subset E$, such that $T_i x = 0$ for every $x \in E_i^\perp$, $i \in \{1, \dots, n\}$.

Lemma 4.3. *Let E, F be vector lattices, E atomless and with the projection property, F be Dedekind complete, and $\{T_1, \dots, T_n\} \subset \mathcal{U}(E, F)$ be a family of order narrow operators with pairwise disjoint supports. Then $T = \sum_{i=1}^n T_i$ is an order narrow operator as well.*

Proof. Fix an arbitrary element $e \in E$. Let ρ_i , be a band projection to the band E_i , $i \in \{1, \dots, n\}$, and $\zeta = Id - \bigvee_{i=1}^n \rho_i$. Then we may write $e = h \sqcup \bigcup_{i=1}^n e_i$, where $e_i = \rho_i e$, $i \in \{1, \dots, n\}$ and $h = \zeta e$. By our assumption for every e_i , $i \in \{1, \dots, n\}$ there exists a net of decompositions $e_i = e_{i1}^\alpha \sqcup e_{i2}^\alpha$ such that

$(T_i(e_{i1}^\alpha) - T_i(e_{i2}^\alpha)) \xrightarrow{(o)} 0$. Let $f_\alpha = \bigsqcup_{i=1}^n e_{i1}^\alpha$ and $g_\alpha = \bigsqcup_{i=1}^n e_{i2}^\alpha$. Now we have

$$\begin{aligned} |T(h + f_\alpha) - T(g_\alpha)| &= \left| \sum_{i=1}^n T_i\left(h \sqcup \bigsqcup_{j=1}^n e_{j1}^\alpha\right) - \sum_{i=1}^n T_i\left(\bigsqcup_{j=1}^n e_{j2}^\alpha\right) \right| = \\ &= \left| \sum_{i=1}^n T_i(e_{i1}^\alpha) - \sum_{i=1}^n T_i(e_{i2}^\alpha) \right| = \left| \sum_{i=1}^n (T_i(e_{i1}^\alpha) - T_i(e_{i2}^\alpha)) \right| \leq \\ &= \sum_{i=1}^n |T_i(e_{i1}^\alpha) - T_i(e_{i2}^\alpha)| \xrightarrow{(o)} 0. \end{aligned}$$

Thus $(h \sqcup f_\alpha) \sqcup g_\alpha = e$ is the desired net of decompositions. ■

Lemma 4.4. *Let E, F be the same as in the Theorem 4.2, $x_1, x_2 \in E$ and $x_1 \perp x_2$. Then $\pi^{x_1+x_2}T = \pi^{x_1}T + \pi^{x_2}T$ for every $T \in \mathcal{U}_+(E, F)$.*

Proof. Fix an arbitrary element $x \in E$. Then for every $y \in \mathcal{F}_x$ so that $y \sqsubseteq (x_1 + x_2)$, we have $y = y_1 \sqcup y_2$, $y_i \sqsubseteq x_i$, $i \in \{1, 2\}$ and the following inequalities hold

$$\begin{aligned} Ty &= Ty_1 + Ty_2 \leq \pi^{x_1}Tx + \pi^{x_2}Tx; \\ \pi^{x_1+x_2}Tx &\leq \pi^{x_1}Tx + \pi^{x_2}Tx. \end{aligned}$$

On the other hand for every $y_i \sqsubseteq x_i$, $y_i \sqsubseteq x$, $i \in \{1, 2\}$ we may write

$$\begin{aligned} Ty_1 + Ty_2 &= T(y_1 + y_2) \leq \pi^{x_1+x_2}Tx; \\ \pi^{x_1}Tx + \pi^{x_2}Tx &\leq \pi^{x_1+x_2}Tx. \end{aligned}$$

■

Proof of Theorem 4.2. Let $T \in \mathcal{U}_+(E, F)$ be an order narrow operator, and $x \in E$. Firstly we prove that operator $\rho\pi^xT$ is also order narrow. Fix an arbitrary element $e \in E$. By our assumption there exists a net of decompositions $e = f_\alpha \sqcup g_\alpha$ such that $|T(f_\alpha) - T(g_\alpha)| \leq \eta_\alpha$, $(\eta_\alpha) \subset F_+$ and $(\eta_\alpha) \downarrow 0$. Remark that $D = \{y \sqsubseteq e : y \in \mathcal{F}_x\}$ is a directed set and by definition of the operator π^xT there exists a net $(y_\alpha) \subset D$ so that

$$|\pi^xTe - Ty_\alpha| = |\pi^xTe - \pi^xTy_\alpha| = |\pi^xT(e - y_\alpha)| \leq \xi_\alpha$$

for some decreasing net $(\xi_\alpha) \subset F_+$, $\inf_\alpha \xi_\alpha = 0$. By our assumption there exists a net of decompositions $y_\alpha = f_\alpha \sqcup g_\alpha$ such that $|T(f_\alpha) - T(g_\alpha)| \leq \eta_\alpha$, $(\eta_\alpha) \subset F_+$ and $(\eta_\alpha) \downarrow 0$. Then we may write

$$\begin{aligned} |\pi^xT((e - y_\alpha) \sqcup f_\alpha) - \pi^xT(g_\alpha)| &= \\ |\pi^xT(e - y_\alpha) + \pi^xTf_\alpha - \pi^xTg_\alpha| &= \\ |\pi^xT(e - y_\alpha) + Tf_\alpha - Tg_\alpha| &\leq \\ |\pi^xT(e - y_\alpha)| + |Tf_\alpha - Tg_\alpha| &\leq \xi_\alpha + \eta_\alpha \xrightarrow{(o)} 0. \end{aligned}$$

So $((e - y_\alpha) \sqcup f_\alpha) \sqcup g_\alpha = e$ is a desired net of decompositions. It is clear that operator $\rho\pi^x T$ is order narrow as well. Secondly, take the operator $R = \sum_{i=1}^n \rho_i \pi^{x_i} T$, where x_1, \dots, x_n are fragments of a some element $x \in E$ and ρ_1, \dots, ρ_n are mutually disjoint. By Lemma 4.4, we may assume that all fragments x_1, \dots, x_n are mutually disjoint. Then applying Lemma 4.3 we prove that R is an order narrow operator. Now, let $(R_\xi)_{\xi \in \Xi} \subset \mathcal{U}_+(E, F)$ be an increasing (decreasing) net of order narrow operators and $S = \sup_{\xi} R_\xi$ ($S = \inf_{\xi} R_\xi$). This meant that there exists a decreasing net $(G_\xi)_{\xi \in \Xi} \subset \mathcal{U}_+(E, F)$, so that $\inf_{\xi} G_\xi = 0$ and

$$|Se - R_\xi e| = |(S - R_\xi)e| \leq |S - R_\xi|e \leq G_\xi e$$

for every $e \in E$. Let us show that S is also order narrow. Indeed, fix an arbitrary element $e \in E$ and write

$$\begin{aligned} |Sf_\alpha - Sg_\alpha| &= |Sf_\alpha - R_\xi f_\alpha + R_\xi f_\alpha - R_\xi g_\alpha + R_\xi g_\alpha - Sg_\alpha| \leq \\ &= |Sf_\alpha - R_\xi f_\alpha| + |R_\xi f_\alpha - R_\xi g_\alpha| + |Sg_\alpha - R_\xi g_\alpha| \leq \\ &= G_\xi f_\alpha + |R_\xi f_\alpha - R_\xi g_\alpha| + G_\xi g_\alpha \leq \\ &= G_\xi e + |R_\xi f_\alpha - R_\xi g_\alpha| + G_\xi e \xrightarrow{(o)} 0. \end{aligned}$$

By Theorem 3.14 we have that $\mathcal{F}_T = \mathfrak{A}_T^{\uparrow\downarrow\uparrow}$ and applying this equality we obtain that every fragment of an order narrow operator T is also order narrow. Finally take an arbitrary operator $S \in \mathcal{U}(E, F)$, so that $0 \leq S \leq kT$, $k \in \mathbb{R}_+$. By Theorem 4.1 there exists a sequence R_n of T -step positive abstract Uryson operators $R_n = \sum_{i=1}^{k_n} \lambda_i C_i$, where $\lambda_i > 0$ for $i \in \{1, \dots, n\}$ and the operators C_1, \dots, C_{k_n} are pairwise disjoint fragments of T such that so that $|S(e) - R_n(e)| \leq \frac{1}{n}T(e)$ for every $e \in E$. Dividing by $\max\{\lambda_i : i = 1, \dots, k_n\}$ we may assume that $\lambda_i \leq 1$ for every $i \in \{1, \dots, k_n\}$ and therefore $0 \leq R_n = \sum_{i=1}^{k_n} \lambda_i C_i = \bigvee_{i=1}^{k_n} \lambda_i C_i \leq T$ is a fragment of the operator T for every $n \in \mathbb{N}$. Thus R_n is an order narrow operator for every $n \in \mathbb{N}$. Finally, using the same arguments as above, we obtain that S is an order narrow operator. ■

Remark that for linear positive operators the similar theorem was proved by Flores and Ruiz in [7].

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